

ON AUTOMORPHISMS OF THE AFFINE CREMONA GROUP

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ABSTRACT. We show that every automorphism of the group $\mathcal{G}_n := \text{Aut}(\mathbb{A}^n)$ of polynomial automorphisms of complex affine n -space $\mathbb{A}^n = \mathbb{C}^n$ is inner up to field automorphisms when restricted to the subgroup $T\mathcal{G}_n$ of tame automorphisms. This generalizes a result of JULIE DESERTI who proved this in dimension $n = 2$ where all automorphisms are tame: $T\mathcal{G}_2 = \mathcal{G}_2$.

1. **Notation.** Let $\mathcal{G}_n := \text{Aut}(\mathbb{A}^n)$ denote the group of polynomial automorphisms of complex affine n -space $\mathbb{A}^n = \mathbb{C}^n$. For an automorphism \mathbf{g} we use the notation $\mathbf{g} = (g_1, g_2, \dots, g_n)$ if

$$\mathbf{g}(a) = (g_1(a_1, \dots, a_n), \dots, g_n(a_1, \dots, a_n)) \quad \text{for } a = (a_1, \dots, a_n) \in \mathbb{A}^n$$

where $g_1, \dots, g_n \in \mathbb{C}[x_1, \dots, x_n]$. Moreover, we define the degree of \mathbf{g} by $\deg \mathbf{g} := \max(\deg g_1, \dots, \deg g_n)$. The product of two automorphisms is denoted by $\mathbf{f} \circ \mathbf{g}$.

The automorphisms of the form (g_1, \dots, g_n) where $g_i = g_i(x_i, \dots, x_n)$ depends only on x_i, \dots, x_n , form the *Jonquière subgroup* $\mathcal{J}_n \subset \mathcal{G}_n$. Moreover, we have the inclusions $D_n \subset \text{GL}_n \subset \text{Aff}_n \subset \mathcal{G}_n$ where D_n is the group of *diagonal automorphisms* $(a_1 x_1, \dots, a_n x_n)$ and Aff_n the group of *affine transformations* $\mathbf{g} = (g_1, \dots, g_n)$ where all g_i are linear. Aff_n is the semidirect product of GL_n with the commutative unipotent subgroup \mathcal{T}_n of translations. The subgroup $T\mathcal{G}_n \subset \mathcal{G}_n$ generated by \mathcal{J}_n and Aff_n is called the group of *tame* automorphisms.

Main Theorem. *Let θ be an automorphism of \mathcal{G}_n . Then there is an element $\mathbf{g} \in \mathcal{G}_n$ and a field automorphism $\tau: \mathbb{C} \rightarrow \mathbb{C}$ such that*

$$\theta(\mathbf{f}) = \tau(\mathbf{g} \circ \mathbf{f} \circ \mathbf{g}^{-1}) \quad \text{for all tame automorphisms } \mathbf{f} \in T\mathcal{G}_n.$$

After some preparation in the following sections the proof is given in section 7. For $n = 2$ where $T\mathcal{G}_2 = \mathcal{G}_2$ this result is due to JULIE DESERTI [Dés06]. In fact, she proved this for any uncountable field K of characteristic zero. Our methods work for any algebraically closed field of characteristic zero.

2. **Ind-group structure and locally finite automorphisms.** The group \mathcal{G}_n has the structure of an ind-group given by $\mathcal{G}_n = \bigcup_{d \geq 1} (\mathcal{G}_n)_d$ where $(\mathcal{G}_n)_d$ are the automorphisms of degree $\leq d$ (see [Kum02]). Each $(\mathcal{G}_n)_d$ is an affine variety and $(\mathcal{G}_n)_d \subset (\mathcal{G}_n)_{d+1}$ is closed for all d . This defines a topology on \mathcal{G}_n where a subset $X \subset \mathcal{G}_n$ is closed (resp. open) if and only if $X \cap (\mathcal{G}_n)_d$ is closed (resp. open) in $(\mathcal{G}_n)_d$ for all d . All subgroups mentioned above are closed subgroups.

In addition, multiplication $\mathcal{G}_n \times \mathcal{G}_n \rightarrow \mathcal{G}_n$ and inverse $: \mathcal{G}_n \rightarrow \mathcal{G}_n$ are morphisms of ind-varieties where for the latter one has to use the fact due to OFER GABBER

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that $\deg \mathbf{f}^{-1} \leq (\deg \mathbf{f})^{n-1}$ (see [BCW82, (1.5) Theorem]). It follows that for every subgroup $G \subset \mathcal{G}_n$ the closure \bar{G} in \mathcal{G}_n is also a subgroup.

A closed subgroup G contained in some $(\mathcal{G}_n)_d$ is called an *algebraic subgroup*. In fact, such a G is an affine algebraic group which acts faithfully on \mathbb{A}^n , and for every algebraic group H acting on \mathbb{A}^n the image of H in \mathcal{G}_n is an algebraic subgroup.

A subset $X \subset \mathcal{G}_n$ is called *bounded constructible*, if X is a constructible subset of some $(\mathcal{G}_n)_d$.

Lemma 1. *Let $G \subset \mathcal{G}_n$ be a subgroup and let $X \subset G$ be a subset which is dense in G and bounded constructible. Then \bar{G} is an algebraic subgroup, and $\bar{G} = X \circ X$.*

Proof. By assumption $G \subset \bar{X} \subset (\mathcal{G}_n)_d$ for some d and so $\bar{G} = \bar{X}$ is an algebraic subgroup. Moreover, there is a subset $U \subset X$ which is open and dense in \bar{G} . Then $U \circ U = \bar{G}$, and so $\bar{G} = G = X \circ X$. \square

An element $\mathbf{g} \in \mathcal{G}_n$ is called *locally finite* if it induces a locally finite automorphism of the algebra $\mathbb{C}[x_1, \dots, x_n]$ of polynomial functions on \mathbb{A}^n . This is equivalent to the condition that the linear span of $\{(\mathbf{g}^m)^*(f) \mid m \in \mathbb{Z}\}$ is finite dimensional for all $f \in \mathbb{C}[x_1, \dots, x_n]$.

More generally, an action of a group G on an affine variety X is called *locally finite* if the induced action on the coordinate ring $\mathcal{O}(X)$ is locally finite, i.e. for all $f \in \mathcal{O}(X)$ the linear span $\langle Gf \rangle$ is finite dimensional. It is easy to see that the image of G in $\text{Aut}(X)$ is dense in an algebraic group \bar{G} which acts algebraically on X . In fact, one first chooses a finite dimensional G -stable subspace $W \subset \mathcal{O}(X)$ which generates $\mathcal{O}(X)$, and then defines $\bar{G} \subset \text{GL}(W)$ to be the closure of the image of G inside $\text{GL}(W)$.

The next result will be used in the following section. We start again with an action of a group G on an affine variety X and assume that $x_0 \in X$ is a fixed point. Then we obtain a representation $\tau: G \rightarrow \text{GL}(T_{x_0}X)$ on the tangent space at x_0 , given by $\tau(g) := d_{x_0}g$.

Lemma 2. *Let G act faithfully on an irreducible affine variety X . Assume that $x_0 \in X$ is a fixed point and that there is a G -stable decomposition $\mathfrak{m}_{x_0} = V \oplus \mathfrak{m}_{x_0}^2$. Then the tangent representation $\tau: G \rightarrow \text{GL}(T_{x_0}X)$ is faithful.*

Proof. Let $g \in \ker \tau$. Then g acts trivially on V , hence on all powers V^j of V . This implies that the action of g on $\mathcal{O}(X)/\mathfrak{m}_{x_0}^k$ is trivial for all $k \geq 1$. Since $\bigcap_k \mathfrak{m}_{x_0}^k = \{0\}$ the claim follows. \square

We remark that a G -stable decomposition $\mathfrak{m}_{x_0} = V \oplus \mathfrak{m}_{x_0}^2$ like in the lemma above always exists if G is a reductive algebraic group.

3. Tori and centralizers. Define $\mu_k := \{\mathbf{g} \in D_n \mid \mathbf{g}^k = \text{id}\}$. We have $\mu_k \simeq (\mathbb{Z}/k)^n$, and $\mu_\infty := \bigcup_k \mu_k \subset D_n$ is the subgroup of elements of finite order where $\mu_\infty \simeq (\mathbb{Q}/\mathbb{Z})^n$. The next lemma about the centralizer of μ_k is easy.

Lemma 3. *For every $k > 1$ we have $\text{Cent}_{\mathcal{G}_n}(\mu_k) = \text{Cent}_{\text{GL}_n}(\mu_k) = D_n$.*

The following result is crucial for the proof of the main theorem.

Proposition 1. *Let $\mu \subset \mathcal{G}_n$ be a finite subgroup isomorphic to μ_2 . Then $\text{Cent}_{\mathcal{G}_n}(\mu)$ is a diagonalizable algebraic subgroup of \mathcal{G}_n , i.e. isomorphic to a closed subgroup of a torus. Moreover $\dim \text{Cent}_{\mathcal{G}_n}(\mu) \leq n$.*

Proof. We first remark that $\text{Cent}_{\mathcal{G}_n}(\mu)$ is a closed subgroup of \mathcal{G}_n . By Smith Theory (see [Ser09, Theorem 7.5]) we know that the fixed point set $F := (\mathbb{A}^n)^{\mu'}$ of every subgroup $\mu' \subset \mu$ is $\mathbb{Z}/2$ -acyclic, in particular non-empty and connected. We also know that F is smooth and that $T_a F = (T_a \mathbb{A}^n)^{\mu'}$ since μ' is linearly reductive (see [Fog73, Theorem 5.2]). If $a \in (\mathbb{A}^n)^\mu$, then the tangent representation of μ on $T_a \mathbb{A}^n$ is faithful, by Lemma 2 above, and so a is an isolated fixed point. Hence, $(\mathbb{A}^n)^\mu = \{a\}$.

Choose generators $\sigma_1, \dots, \sigma_n$ of μ such that the images in $\text{GL}(T_a \mathbb{A}^n)$ are reflections, i.e. have a single eigenvalue -1 , and set $H_i := (\mathbb{A}^n)^{\sigma_i}$. The tangent representation shows that H_i is a hypersurface, hence defined by an irreducible polynomial $f_i \in \mathbb{C}[x_1, \dots, x_n]$. Moreover, $\sigma_i^*(f_i) = -f_i$ and $\sigma_i^*(f_j) = f_j$ for $j \neq i$. It follows that the linear subspace $V := \mathbb{C}f_1 \oplus \dots \oplus \mathbb{C}f_n \subset \mathbb{C}[x_1, \dots, x_n]$ is μ -stable. In addition, any $g \in G := \text{Cent}_{\mathcal{G}_n}(\mu)$ fixes a and stabilizes all $\mathbb{C}f_i$ and so, by the following Lemma 4 applied to the morphism $\varphi := (f_1, \dots, f_n): \mathbb{A}^n \rightarrow \mathbb{A}^n$, the action of G on \mathbb{A}^n is locally finite. Since G is a closed subgroup of \mathcal{G}_n , it follows that it is an algebraic subgroup of \mathcal{G}_n , and its image in $\text{GL}(V)$ is a closed subgroup contained in a maximal torus, hence a diagonalizable group.

Finally, $\mathfrak{m}_a = V \oplus \mathfrak{m}_a^2$, and thus the homomorphism $G \rightarrow \text{GL}(T_a \mathbb{A}^n)$ is injective, by Lemma 2. Hence the claim. \square

Remark 1. It is not difficult to show that the proposition holds for every finite commutative subgroup μ of rank n . In fact, the proof carries over to subgroups isomorphic to μ_p where p is a prime, and every finite commutative subgroup μ of rank n contains such a group.

Lemma 4. *Let $G \subset \text{Aut}(\mathbb{A}^n)$ be a subgroup and let $\varphi: \mathbb{A}^n \rightarrow X$ be a dominant morphism. Assume that $\varphi^*(\mathcal{O}(X))$ is a G -stable subalgebra and that the induced action of G on X is locally finite. Then the same holds for the action of G on \mathbb{A}^n .*

Proof. Put $A := \varphi^*(\mathcal{O}(X)) \subset \mathbb{C}[x_1, \dots, x_n]$ and denote by $R \subset \mathbb{C}[x_1, \dots, x_n]$ the integral closure of A . We first claim that the action of G on R is locally finite. In fact, let $f \in R$ and let $f^m + a_1 f^{m-1} + \dots + a_m = 0$ be an integral equation of f over A . By assumption, the spaces $\langle Ga_i \rangle$ are all finite dimensional, and so there is a $d \in \mathbb{N}$ such that $\deg ga_i < d$ for all $g \in G$ and all a_i . Since gf satisfies the equation $(gf)^m + (ga_1)(gf)^{m-1} + \dots + (ga_m) = 0$ we get $\deg(gf) < d$ for all $g \in G$, hence the claim.

Therefore, we can assume that X is normal and that $\varphi: \mathbb{A}^n \rightarrow X$ is birational. Choose an open set $U \subset \mathbb{A}^n$ such that $\varphi(U) \subset X$ is open and φ induces an isomorphism $U \xrightarrow{\sim} \varphi(U)$. Define $Y := \bigcup_{g \in G} gU \subset \mathbb{A}^n$. Then the induced morphism $\psi := \varphi|_Y: Y \rightarrow \varphi(Y)$ is G -equivariant and a local isomorphism. Since X is quasi-compact the fibers of ψ are finite, and since ψ is birational and $\varphi(Y)$ normal we get that ψ is a G -equivariant isomorphism.

By assumption, the action of G on X is locally finite, and so G is dense in an algebraic group \bar{G} which acts regularly on X . Clearly, the open set $\varphi(Y)$ is \bar{G} -stable and thus the action of \bar{G} on $\mathcal{O}(\varphi(Y))$ is locally finite. Now the claim follows, because $\mathbb{C}[x_1, \dots, x_n] \subset \mathcal{O}(Y)$ is a G -stable subalgebra. \square

The proposition above has an interesting consequence for the linearization problem for finite group actions on affine 3-space \mathbb{A}^3 . In this case it is known that every faithful action of a non-finite reductive group on \mathbb{A}^3 is linearizable (KRAFT-RUSSELL, see [KR11]).

Corollary 1. *Let $\mu \subset \mathcal{G}_3$ be a commutative subgroup of rank three. If the centralizer of μ is not finite, then μ is conjugate to a subgroup of D_3 .*

4. D_n -stable unipotent subgroups. Recall that every commutative unipotent group U has a natural structure of a \mathbb{C} -vector space, given by the exponential map $\exp: T_e U \xrightarrow{\sim} U$. Thus $\text{Aut}(U) = \text{GL}(U)$ and every action of an algebraic group on U by group automorphisms is given by a linear representation.

A (non-zero) locally nilpotent vector field $\delta = \sum_{i=1}^n h_i \frac{\partial}{\partial x_i}$ defines a (non-trivial) \mathbb{C}^+ -action on \mathbb{A}^n , hence a one-dimensional unipotent subgroup

$$U_\delta = \{(\exp(t\delta)(x_1), \dots, \exp(t\delta)(x_n)) \mid t \in \mathbb{C}^+\} \subseteq \mathcal{G}_n,$$

and $U_\delta = U_{\delta'}$ if and only if δ' is a scalar multiple of δ .

Lemma 5. *Let $U = U_\delta \subset \mathcal{G}_n$ be a one-dimensional unipotent subgroup. Then U_δ is normalized by D_n if and only if δ is of the form $cx^\gamma \frac{\partial}{\partial x_i}$, where*

$$x^\gamma = x_1^{\gamma_1} \cdots x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}} \cdots x_n^{\gamma_n}$$

and $c \in \mathbb{C}^*$. In particular, $U_\delta = \{\delta(s) := (x_1, \dots, x_i + s(cx^\gamma), \dots, x_n) \mid s \in \mathbb{C}\}$, and $\mathbf{d} \circ \delta(s) \circ \mathbf{d}^{-1} = \delta(t^{e_i - \gamma} s)$ for $\mathbf{d} = (t_1 x_1, \dots, t_n x_n) \in D_n$.

Proof. If U_δ is normalized by D_n , then $\mathbf{d} \circ \delta \circ \mathbf{d}^{-1} \in \mathbb{C}^* \delta$ for all $\mathbf{d} \in D_n$. Writing $\delta = \sum_i h_i \frac{\partial}{\partial x_i}$ it follows that each h_i is a monomial of the form $h_i = a_i x^{\beta + e_i}$ for some $\beta \in \mathbb{Z}^n$. If $\beta_i \geq 0$ an induction on m shows that, for all $m \geq 1$, we have

$$\delta^m(x_i) = b_m^{(i)} x^{m\beta + e_i}, \quad \text{where } b_m^{(i)} = a_i \prod_{l=1}^{m-1} (lb + a_i) \quad \text{and } b = \sum_{j=1}^n a_j \beta_j.$$

Assume that $\beta_i \geq 0$ for all i . Since δ is locally nilpotent there is a minimal $m_i \geq 0$ such that $b_{m_i+1}^{(i)} = 0$. This implies $a_i = -m_i b$. Since $\delta \neq 0$, we get

$$0 \neq b = \sum_{i=1}^n a_i \beta_i = -b \sum_{i=1}^n m_i \beta_i,$$

and so $\sum m_i \beta_i = -1$. But this is a contradiction, because $m_i, \beta_i \geq 0$ for all i . Therefore $a_i x^{\beta + e_i} \neq 0$ implies that $\beta_j \geq 0$ for all $j \neq i$, and $\beta_i = -1$. Thus there is only one term in the sum, i.e. $\delta = a_i x^\gamma \frac{\partial}{\partial x_i}$ where $\gamma := \beta + e_i$ has the claimed form. \square

Remark 2. This lemma can also be expressed in the following way: *There is a bijective correspondence between the D_n -stable one-dimensional unipotent subgroups $U \subset \mathcal{G}_n$ and the characters of D_n of the form $\lambda = \sum_j \lambda_j \varepsilon_j$ where one λ_i equals 1 and the others are ≤ 0 .* We will denote this set of characters by $X_u(D_n)$:

$$X_u(D_n) := \{\lambda = \sum \lambda_j \varepsilon_j \mid \exists i \text{ such that } \lambda_i = 1 \text{ and } \lambda_j \leq 0 \text{ for } j \neq i\}.$$

If $\lambda \in X_u(D_n)$, then U_λ denotes the corresponding one-dimensional unipotent subgroup normalized by D_n .

Remark 3. In [Lie11, Theorem 1] ALVARO LIENDO shows that the locally nilpotent derivations normalized by the torus $D'_n := D_n \cap \text{SL}_n$ have exactly the same form.

Lemma 6. *The subgroup \mathcal{T}_n of translations is the only commutative unipotent subgroup normalized by GL_n .*

Proof. If $U \subset \mathcal{G}_n$ is a commutative unipotent subgroup normalized by GL_n , then all the weights of the representation of GL_n on $T_e U \simeq U$ must belong to $X_u(D_n)$. The dominant weights of GL_n are $\sum_i \lambda_i \varepsilon_i$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, and only those of the form $\lambda = \varepsilon_1 + \sum_{i>1} \lambda_i \varepsilon_i$ where $0 \geq \lambda_2 \geq \dots \geq \lambda_n$ occur in $X_u(D_n)$. If $\lambda \neq \varepsilon_1$, i.e. $\lambda = \varepsilon_1 + \lambda_k \varepsilon_k + \lambda_{k+1} \varepsilon_{k+1} + \dots$ where $\lambda_k < 0$, then the weight $\lambda' := (\lambda_k + 1) \varepsilon_k + \lambda_{k+1} \varepsilon_{k+1} + \dots$ is dominant and $\lambda' \prec \lambda$. Therefore λ' appears in the irreducible representation of GL_n of highest weight λ , but $\lambda' \notin X_u(D_n)$. Thus U and \mathcal{T}_n are isomorphic as GL_n -modules, hence contain the same D_n -stable one-dimensional unipotent subgroups, and so $U = \mathcal{T}_n$. \square

5. Maximal tori. It is clear that $D_n \subset \mathcal{G}_n$ is a maximal commutative subgroup of \mathcal{G}_n since it coincides with its centralizer, see Lemma 3. Moreover, BIAŁYŃICKI-BIRULA proved in [BB66] that a faithful action of an n -dimensional torus on \mathbb{A}^n is linearizable (cf. [KS92, Chap. I.2.4, Theorem 5]). Thus we have the following result.

Lemma 7. *D_n is a maximal commutative subgroup of \mathcal{G}_n . Moreover, every algebraic subgroup of \mathcal{G}_n , which is isomorphic to D_n is conjugate to D_n .*

Now let $G \subset \mathcal{G}_n$ be an algebraic subgroup which is normalized by D_n . Then the non-zero weights of the representation of D_n on the Lie algebra $\mathrm{Lie} G$ belong to $X_u(D_n)$, and the weight spaces are one-dimensional. It follows that the non-zero weight spaces of $\mathrm{Lie} G$ are in bijective correspondence with the D_n -stable one-dimensional unipotent subgroups of G .

Lemma 8. *Let $G \subset \mathcal{G}_n$ be an algebraic subgroup normalized by a torus $D \subset \mathcal{G}_n$ of dimension n , let U_1, \dots, U_r be the D -stable one-dimensional unipotent subgroups of G , and put $X := U_1 \circ \dots \circ U_r \subset G$.*

- (a) *If G is unipotent, then $G = X \circ X$ and $\dim G = r$.*
- (b) *If $D \subset G$, then $G^0 = D \circ X \circ D \circ X$ and $\dim G = r + n$.*

Proof. (a) The canonical map $U_1 \times \dots \times U_r \rightarrow G$ is dominant, and so $X \subset G$ is constructible and dense. Thus $X \circ X = G$, by Lemma 3, and $\dim G = \dim \mathrm{Lie} G = r$.

(b) Similarly, we see that $D \circ X \subset G^0$ is constructible and dense, and therefore $D \circ X \circ D \circ X = G^0$, and $\dim G = \dim \mathrm{Lie} G = \dim \mathrm{Lie} D + r$. \square

6. Images of algebraic subgroups. The next two propositions are crucial for the proof of our main theorem.

Proposition 2. *Let θ be an automorphism of \mathcal{G}_n . Then*

- (a) *$D := \theta(D_n)$ is a torus of dimension n , conjugate to D_n .*
- (b) *If U is a D_n -stable unipotent subgroup, then $\theta(U)$ is a D -stable unipotent subgroup of the same dimension.*
- (c) *$\mathcal{T} := \theta(\mathcal{T}_n)$ is a commutative unipotent subgroup of dimension n , normalized by D , and the representation of D on \mathcal{T} is faithful.*

Proof. (a) We have $D_n = \mathrm{Cent}_{\mathcal{G}_n}(\mu_2)$, by Lemma 3, and thus $D = \theta(D_n) = \mathrm{Cent}_{\mathcal{G}_n}(\theta(\mu_2))$. Proposition 1 implies that D is a diagonalizable algebraic subgroup with $\dim D \leq n$, hence $D = D^0 \times F$ for some finite group F . If k is prime to the order of F , then $\theta(\mu_k) \subset D^0$ and so $\dim D^0 = n$, because $\mu_k \simeq (\mathbb{Z}/k)^n$. Hence $D = D^0$ is an n -dimensional torus which is conjugate to D_n , by Lemma 7.

(b) First assume that $\dim U = 1$. Then U consists of two D_n -orbits, $O := U \setminus \{\mathrm{id}\}$ and $\{\mathrm{id}\}$. It follows that $\theta(U)$ consists of the two D -orbits $\theta(O)$ and $\{\mathrm{id}\}$, and so

$\theta(U)$ is bounded constructible and thus a commutative algebraic group normalized by D . Since it does not contain elements of finite order it is unipotent, and since it consists of only two D -orbits it is of dimension 1.

Now let U be arbitrary, $\dim U = r$, and let U_1, \dots, U_r be the different D_n -stable one-dimensional unipotent subgroups of U . Then $X := U_1 \circ U_2 \circ \dots \circ U_r \subset U$ is dense and constructible and $U = X \circ X$, by Lemma 8(a). Applying θ implies that $\theta(X) = \theta(U_1) \circ \dots \circ \theta(U_r)$ is bounded constructible and connected, as well as $\theta(U) = \theta(X) \circ \theta(X)$, and thus $\theta(U)$ is a connected algebraic subgroup of \mathcal{G}_n normalized by D . Since every element of $\theta(U)$ has infinite order, $\theta(U)$ must be unipotent. Moreover, $\dim \theta(U) \geq r$, since $\theta(U)$ contains the D -stable one-dimensional unipotent subgroups $\theta(U_i)$, $i = 1, \dots, r$. The same argument applied to θ^{-1} finally gives $\dim \theta(U) = r$.

(c) This statement follows from (b) and the fact that \mathcal{T}_n contains a dense D_n -orbit with trivial stabilizer. \square

The same arguments, this time using Lemma 8(b), gives the next result.

Proposition 3. *Let θ be an automorphism of \mathcal{G}_n and let $G \subset \mathcal{G}_n$ be an algebraic subgroup which contains a torus D of dimension n .*

- (a) *The image $\theta(G)$ is an algebraic subgroup of \mathcal{G}_n of the same dimension $\dim G$.*
- (b) *We have $\theta(G^0) = \theta(G)^0$. In particular, $\theta(G)$ is connected if G is connected.*
- (c) *If G is reductive, then so is $\theta(G)$, and then $\theta(G)$ is conjugate to a closed subgroup of GL_n .*

Proof. As above, let U_1, \dots, U_r be the different D -stable one-dimensional unipotent subgroups of G , and put $X := U_1 \circ \dots \circ U_r$. Then $D \circ X$ is constructible in G^0 , and $D \circ X \circ D \circ X = G^0$, by Lemma 8(b). Applying θ we see that $\theta(D \circ X \circ D \circ X) = \theta(D) \circ \theta(X) \circ \theta(D) \circ \theta(X)$ is bounded constructible and connected, and so $\theta(G^0)$ is a connected algebraic subgroup of \mathcal{G}_n , of finite index in $\theta(G)$. Since the $\theta(U_i)$ are different $\theta(D)$ -stable one-dimensional unipotent subgroups of $\theta(G)$ we have $\dim \theta(G) \geq \dim \theta(D) + r = \dim G$. Using θ^{-1} we get equality. This proves (a) and (b).

For (c) we remark that if G contains a normal unipotent subgroup U , then $\theta(U)$ is a normal unipotent subgroup of $\theta(G)$. Moreover, a reductive subgroup G containing a torus of dimension n has no non-constant invariants, and so G is linearizable (see [KP85, Proposition 5.1]). \square

7. Proof of the Main Theorem. Let θ be an automorphism of \mathcal{G}_n . It follows from Proposition 3 that there is a $\mathbf{g} \in \mathcal{G}_n$ such that $\mathbf{g} \circ \theta(\mathrm{GL}_n) \circ \mathbf{g}^{-1} \subset \mathrm{GL}_n$. Therefore we can assume that $\theta(\mathrm{GL}_n) = \mathrm{GL}_n$. The subgroup \mathcal{T}_n of translations is the only commutative unipotent subgroup normalized by GL_n , by Lemma 6. Therefore, $\theta(\mathcal{T}_n) = \mathcal{T}_n$ and so $\theta(\mathrm{Aff}_n) = \mathrm{Aff}_n$. Now the theorem follows from the next proposition. \square

Proposition 4. (a) *Every automorphism θ of Aff_n with $\theta(\mathrm{GL}_n) = \mathrm{GL}_n$ and $\theta(\mathcal{T}_n) = \mathcal{T}_n$ is of the form $\theta(\mathbf{f}) = \tau(\mathbf{g} \circ \mathbf{f} \circ \mathbf{g}^{-1})$ where $\mathbf{g} \in \mathrm{GL}_n$ and τ is an automorphism of the field \mathbb{C} .*

(b) *If θ is an automorphism of \mathcal{G}_n such that $\theta|_{\mathrm{Aff}_n} = \mathrm{Id}_{\mathrm{Aff}_n}$, then $\theta|_{\mathcal{T}_n} = \mathrm{Id}_{\mathcal{T}_n}$.*

Proof. (a) It is enough to prove $\theta(f) = \mathbf{g} \circ \tau(\mathbf{f}) \circ \mathbf{g}^{-1}$ for some $\mathbf{g} \in \mathrm{GL}_n$ and some automorphism $\tau: \mathbb{C} \rightarrow \mathbb{C}$ of the field \mathbb{C} . Let $\mathbb{C}^* = Z \subseteq \mathrm{GL}_n$ be the center of GL_n

and define $\theta_0 := \theta|_Z : Z \rightarrow Z$, $\theta_1 := \theta|_{\mathcal{T}_n} : \mathcal{T}_n \rightarrow \mathcal{T}_n$. It follows that θ_0 and θ_1 are abstract group homomorphisms of \mathbb{C}^* and \mathcal{T}_n respectively, and for all $c \in \mathbb{C}^*$, $\mathbf{t} \in \mathcal{T}_n$

$$\theta_1(c \cdot \mathbf{t}) = \theta_0(c) \cdot \theta_1(\mathbf{t}),$$

where “ \cdot ” denotes scalar multiplication. This implies that $\tau : \mathbb{C} \rightarrow \mathbb{C}$ defined by $\tau|_{\mathbb{C}^*} = \theta_0$, $\tau(0) = 0$, is an automorphism of the field \mathbb{C} . Hence we can assume $\theta_0 = \text{id}_{\mathbb{C}^*}$ and therefore θ_1 is linear. Considering θ_1 as an element of GL_n we have $\theta_1(\mathbf{t}) = \theta_1 \circ \mathbf{t} \circ \theta_1^{-1}$, and thus we can assume that $\theta_1 = \text{id}_{\mathcal{T}_n}$. But this implies that $\theta(\mathbf{g}) = \mathbf{g}$ for all $\mathbf{g} \in \text{GL}_n$, because

$$\mathbf{g} \circ \mathbf{t} \circ \mathbf{g}^{-1} = \theta(\mathbf{g} \circ \mathbf{t} \circ \mathbf{g}^{-1}) = \theta(\mathbf{g}) \circ \mathbf{t} \circ \theta(\mathbf{g})^{-1}$$

for all $\mathbf{t} \in \mathcal{T}_n$.

(b) Let $U \subset \mathcal{G}_n$ be a one-dimensional unipotent D_n -stable subgroup. We first claim that $\theta(U) = U$ and that $\theta|_U$ is linear. In fact, $U' := \theta(U)$ is a one-dimensional unipotent D_n -stable subgroup, by Proposition 2(b), and the characters λ and λ' associated to U and U' (see Remark 2) have the same kernel, because

$$(*) \quad \theta(\lambda(\mathbf{d}) \cdot u) = \theta(\mathbf{d} \circ u \circ \mathbf{d}^{-1}) = \mathbf{d} \circ \theta(u) \circ \mathbf{d}^{-1} = \lambda'(\mathbf{d}) \cdot \theta(u) \quad \text{for } \mathbf{d} \in D_n, u \in U.$$

Hence $\lambda = \pm \lambda'$. If $\lambda = -\lambda'$, then $U \subseteq \text{GL}_n$ and so $U' = U$, since $\theta|_{\text{GL}_n} = \text{Id}_{\text{GL}_n}$, hence a contradiction. Thus $\lambda = \lambda'$, and so $U = U'$ and $(*)$ shows that $\theta|_U$ is linear, proving our claim.

As a consequence, $\theta|_{U_\lambda} = a_\lambda \text{Id}_{U_\lambda}$ for all $\lambda \in X_u(D_n)$, with suitable $a_\lambda \in \mathbb{C}^*$. If $\lambda_i = 1$ put $\gamma_i := 0$ and $\gamma_j := -\lambda_j$. Then $\mathbf{f} = (x_1, \dots, x_i + x^\gamma, \dots, x_n) \in U_\lambda$, see Lemma 5. Conjugation with the translation $\mathbf{t} : x \mapsto x - \sum_{j \neq i} e_j$ gives

$$\mathbf{t} \circ \mathbf{f} \circ \mathbf{t}^{-1} = (x_1, \dots, x_i + h_\gamma, \dots, x_n) \text{ where } h_\gamma := (x_1 + 1)^{\gamma_1} (x_2 + 1)^{\gamma_2} \cdots (x_n + 1)^{\gamma_n}.$$

Now we apply θ to get $\theta(\mathbf{t} \circ \mathbf{f} \circ \mathbf{t}^{-1}) = \mathbf{t} \circ \theta(\mathbf{f}) \circ \mathbf{t}^{-1}$. Since all the monomials $x^{\gamma'}$ with $\gamma' \leq \gamma$ appear in h_γ it follows that the corresponding coefficients $a_{\lambda'}$ must all be equal. In particular, $a_\lambda = a_{\varepsilon_i} = 1$ since $U_{\varepsilon_i} \subset \mathcal{T}_n$. This shows that $\theta|_{\mathcal{T}_n} = \text{Id}_{\mathcal{T}_n}$. \square

REFERENCES

- [BB66] A. Białynicki-Birula, *Remarks on the action of an algebraic torus on k^n* , Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **14** (1966), 177–181.
- [BCW82] Hyman Bass, Edwin H. Connell, and David Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 2, 287–330.
- [Dés06] Julie Déserti, *Sur le groupe des automorphismes polynomiaux du plan affine*, J. Algebra **297** (2006), no. 2, 584–599.
- [Fog73] John Fogarty, *Fixed point schemes*, Amer. J. Math. **95** (1973), 35–51.
- [KP85] Hanspeter Kraft and Vladimir L. Popov, *Semisimple group actions on the three-dimensional affine space are linear*, Comment. Math. Helv. **60** (1985), no. 3, 466–479.
- [KR11] Hanspeter Kraft and Peter Russell, *Families of group actions, generic isotrivality, and linearization*, Preprint 2011.
- [KS92] Hanspeter Kraft and Gerald W. Schwarz, *Reductive group actions with one-dimensional quotient*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 76, 1–97.
- [Kum02] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.
- [Lie11] Alvaro Liendo, *Roots of the affine cremona group*, Transform. Groups, to appear (2011).

- [Ser09] Jean-Pierre Serre, *How to use finite fields for problems concerning infinite fields*, Arithmetic, geometry, cryptography and coding theory, Contemp. Math., vol. 487, Amer. Math. Soc., Providence, RI, 2009, pp. 183–193.

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